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USING THE FUZZY EMAD-FALIH TRANSFORM TO SOLVE A FIRST-ORDER FUZZY LINEAR SYSTEM IN N-DIMENSION

Abstract: *In this work, a system of fuzzy linear differential equations of first order in n dimensions is solved using the fuzzy Emad-Falih transform. Additionally, provide an applicable example from one of the institutions to support the work.*

Keywords: *fuzzy number; fuzzy Emad-Falih transform; system of fuzzy linear first order differential equation in n -dimensions.*

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1. Introduction:

The study of fuzzy differential equations and fuzzy integral transforms has received a lot of interest from writers in recent years. For the theoretical conclusions required to cope with such integrals and equations, we direct the reader to [1, 2, 3, 4]. Conversely, several scholars have investigated the "fuzzification" of various methodologies that are often used in the crisp situation, and they have created fuzzy versions of various techniques, such as fuzzy Laplace, Sumudu, Emad-Falih, etc. (see to [5, 6, 7, 8] and the references therein). One application of ambiguous mathematical systems is the computation of the costs of specific activities in public or private institutions. These systems assist accountants in providing investors with an approximate future picture by determining the costs of specific activities for the upcoming years based on the relationship between these activities and how they are represented in a fuzzy mathematical system. The time component and the link between costs are ignored by the ABC approach, which was formerly utilized by accountants to compute expenses [9]. Later, the TDABC system emerged, which accounts for time and computes expenses [10]. In order to get the general solution formula for a system of first-order fuzzy linear differential equations in n dimensions, we create the slavery approach in this study. Additionally, we provide an example in dimension three that is solved using this formula.

2. Basic Concepts

A number of vocabulary keys and fundamental concepts are introduced in this section.

Definition (2.1) [5]:

The The mapping $Ʀ_f$: Ʀ → [0,1] is fuzzy number if satisfies

- ꞷ is upper semi-continuous.
- ꞷ is fuzzy convex, i. e., ꞷ(ϐɧ + (1 − ϐ)ʧ) ≥ min{ꞷ(ɧ), ꞷ(ʧ)},for all ɧ , ʧ ∈ Ʀ and ϐ ∈ [0,1].
- ꞷ is normal i. e., ∃ $ɧ_0$ ∈ Ʀ for which ꞷ(ɧ) = 1.
- Supp (ꞷ) = { ɧ ∈ Ʀ;ꞷ(ɧ) > 0 }, and cl(Supp (ꞷ)) is compact.

Definition (2.2) [11]:

Assume $\natural, \rho \in \mathcal{R}_f$. There is $\pi \in \varpi$ such that $\natural = \rho \oplus \pi$ then π is known the H-differential of $\natural$ *and* ρ and it is represented by $\natural \ominus \rho$. In this paper, the sign " $\ominus$ " always stands for H-difference, and also note that

$\ominus \neq \ominus_h$ *and* $\natural \ominus \rho \neq \natural + (-1)\rho$, where $\mathcal{R}_f$ be the set of all fuzzy number on $\mathcal{R}$

Definition (2.3) [12]:

A parametrically ordered pair is a fuzzy number $(\underline{G}, \overline{G})$ of functions $\overline{G}(\beta)$, $\underline{G}(\beta)$,

$\beta \in [0,1]$, which satisfies

- $\underline{G}(\beta)$ is a 0 continuous right non-decreasing bound and (0,1] left-continuous function.
- $\overline{G}(\beta)$ is a 0 continuous, non-increasing bounded right and (0,1] left-continuous function.
- $\underline{G}(\beta) \leq \overline{G}(\beta)$, $\beta \in [0,1]$.

Theorem (2.1) [13]:

Let $\mathcal{U}(\mathfrak{h})$ a function with fuzzy values on $[e, \infty)$ embodied by $((\underline{\mathcal{U}}(\mathfrak{h}, \beta), \overline{\mathcal{U}}(\mathfrak{h}, \beta))$ .For any fixed $\beta \in [0,1]$ let $(\underline{\mathcal{U}}(\mathfrak{h}, \beta), \overline{\mathcal{U}}(\mathfrak{h}, \beta)$ are Riemann-integrals on $[e, r]$. For every $r \geq$e, if there are two positive functions $\underline{\theta}(\beta)$ and $\overline{\theta}(\beta)$ such that $\int_e^r |\underline{\mathcal{U}}(\mathfrak{h}, \beta)| d\mathfrak{h} \leq \underline{\theta}(\beta)$ *and* $\int_e^r |\overline{\mathcal{U}}(\mathfrak{h}, \beta)| d\mathfrak{h} \leq \overline{\theta}(\beta)$ for every r $\geq$e, then $\mathcal{U}(\mathfrak{h})$ is considered to be inappropriately hazy. The function of Riemann-Liouville integrals on [e,∞), i.e. $\int_e^\infty \mathcal{U}(\mathfrak{h})\, d\mathfrak{h} = [\int_e^\infty (\underline{\mathcal{U}}(\mathfrak{h}, \beta) d\mathfrak{h}, \int_e^\infty \overline{\mathcal{U}}(\mathfrak{h}, \beta) d\mathfrak{h}]$.

Definition (2.4) [14]:

A function $\mathcal{U}: (e, r): \longrightarrow \mathcal{R}_{\mathcal{F}}$ and $\mathfrak{h}_0 \in (e, r)$. If there is an element $\mathcal{U}'(\mathfrak{h}_0) \in \mathcal{R}_{\mathcal{F}}$, we claim that a mapping $\mathcal{U}$ is highly generalized differentiable at $\mathfrak{h}_0$, such that:

- $\forall\, \eta > 0$ that is sufficiently small, there exist $\mathcal{U}(\mathfrak{h}_0 + \eta) \ominus \mathcal{U}(\mathfrak{h}_0), \mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 - \eta)$,

where $\lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0 + \eta) \ominus \mathcal{U}(\mathfrak{h}_0)}{\eta} = \lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 - \eta)}{\eta} = \mathcal{U}'(\mathfrak{h}_0)$ or

- $\forall\, \eta > 0$ that is sufficiently small, there exist $\mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 + \eta), \mathcal{U}(\mathfrak{h}_0 - \eta) \ominus \mathcal{U}(\mathfrak{h}_0)$

where $\lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 + \eta)}{-\eta} = \lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0 - \eta) \ominus \mathcal{U}(\mathfrak{h}_0)}{-\eta} = \mathcal{U}'(\mathfrak{h}_0)$ or

- $\forall\, \eta > 0$ that is sufficiently small, there exist $\mathcal{U}(\mathfrak{h}_0 + \eta) \ominus \mathcal{U}(\mathfrak{h}_0), \mathcal{U}(\mathfrak{h}_0 - \eta) \ominus \mathcal{U}(\mathfrak{h}_0)$

where $\lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0 + \eta) \ominus \mathcal{U}(\mathfrak{h}_0)}{\eta} = \lim\limits_{\tau \to 0} \frac{\mathcal{U}(\mathfrak{h}_0 - \eta) \ominus \mathcal{U}(\mathfrak{h}_0)}{-\eta} = \mathcal{U}'(\mathfrak{h}_0)$ or

- $\forall\, \eta > 0$ that is sufficiently small, there exist $\mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 + \eta), \mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 - \eta)$

where $\lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 + \eta)}{-\eta} = \lim\limits_{\eta \to 0} \frac{\mathcal{U}(\mathfrak{h}_0) \ominus \mathcal{U}(\mathfrak{h}_0 - \eta)}{\eta} = \mathcal{U}'(\mathfrak{h}_0)$.

Theorem (2.2) [5]:

Let $\mathcal{U}(\mathfrak{h}): [e, r] \longrightarrow \mathcal{R}_f$ be a function and represent $\mathcal{U}(\mathfrak{h}) = ((\underline{\mathcal{U}}(\mathfrak{h}, \beta), \overline{\mathcal{U}}(\mathfrak{h}, \beta))$ in each case for $\beta \in [0,1]$. Then:

- If $\mathcal{U}(\mathfrak{h})$ is differentiable in form i , then $(\underline{\mathcal{U}}(\mathfrak{h}, \beta)$ and $\overline{\mathcal{U}}(\mathfrak{h}, \beta)$ are differentiable functions and

$\mathcal{U}'(\mathfrak{h}) = (\underline{\mathcal{U}}'(\mathfrak{h}, \beta), \overline{\mathcal{U}}'(\mathfrak{h}, \beta))$.

➢ If $\mathfrak{w}(\mathfrak{h})$ is differentiable in form ii, then $(\underline{\mathfrak{w}}(\mathfrak{h},\beta)$ and $\overline{\mathfrak{w}}(\mathfrak{h},\beta)$ are differentiable functions and $\mathfrak{w}'(\mathfrak{h}) = (\overline{\mathfrak{w}}'(\mathfrak{h},\beta), \underline{\mathfrak{w}}'(\mathfrak{h},\beta))$.

Definition (2.5) [7]:

Let $\mathfrak{w}(\mathfrak{h})$ be a continuous function with fuzzy values. Suppose that $\frac{1}{r}\mathfrak{w}(\mathfrak{h})\, e^{-r^2\mathfrak{h}}$ is a fuzzy integrable at Rimann that is improper on $[0,\infty)$, then $\frac{1}{r}\int_0^\infty \mathfrak{w}(\mathfrak{h})\, e^{-r^2\chi}\, d\chi$ is referred to as the fuzzy Emad-Falih transform and is

$$\widehat{\mathbb{EF}}[\mathfrak{w}(\mathfrak{h})] = \frac{1}{r}\int_0^\infty \mathfrak{w}(\mathfrak{h})\, e^{-r^2\mathfrak{h}}\, d\mathfrak{h}\ , (r > 0 \text{ and integer})$$

$$\frac{1}{r}\int_0^\infty \mathfrak{w}(\mathfrak{h}) e^{-r^2\mathfrak{h}}\, d\mathfrak{h} = \left(\frac{1}{r}\int_0^\infty \underline{\mathfrak{w}}(\mathfrak{h},\beta)\, e^{-r^2\mathfrak{h}}\, d\mathfrak{h}\, , \frac{1}{r}\int_0^\infty \overline{\mathfrak{w}}(\mathfrak{h},\beta)\, e^{-r^2\mathfrak{h}}\, d\mathfrak{h}\right).$$

Applying the traditional Emad-Falih transform definition, we obtain:

$\mathbb{EF}[\underline{\mathfrak{w}}(\mathfrak{h},\beta)] = \frac{1}{r}\int_0^\infty \underline{\mathfrak{w}}(\mathfrak{h},\beta) e^{-r^2\mathfrak{h}}\, d\mathfrak{h}$ *and* $\mathbb{EF}[\overline{\mathfrak{w}}(\mathfrak{h},\beta)] = \frac{1}{r}\int_0^\infty \overline{\mathfrak{w}}(\mathfrak{h},\beta) e^{-r^2\mathfrak{h}}\, d\mathfrak{h}$, then:

$$\widehat{\mathbb{EF}}[\mathfrak{w}(\mathfrak{h})] = (\mathbb{EF}[\underline{\mathfrak{w}}(\mathfrak{h},\beta)], \mathbb{EF}[\overline{\mathfrak{w}}(\mathfrak{h},\beta)])$$

Theorem (2.3) [7]:

Let $\mathfrak{w}(\mathfrak{h})$ is the primitive of $\mathfrak{w}'(\mathfrak{h})$ on $[0,\infty)$ and $\mathfrak{w}(\mathfrak{h})$ is a fuzzy-valued function that is integrable, then:

$\mathfrak{w}(\mathfrak{h})$is (i)-differentiable then $\widehat{\mathbb{EF}}[\mathfrak{w}'(\mathfrak{h})] = r^2\widehat{\mathbb{EF}}[\mathfrak{w}(\mathfrak{h})] \ominus \frac{1}{r}\mathfrak{w}(0)$.

$\mathfrak{w}(\mathfrak{h})$is (ii)-differentiable then $\widehat{\mathbb{EF}}[\mathfrak{w}'(\mathfrak{h})] = \left(-\frac{1}{r}\mathfrak{w}(0)\right) \ominus \left(-r^2\widehat{\mathbb{EF}}[\mathfrak{w}(\mathfrak{h})]\right)$

3. Linear Differential Equations in n-Dimensional Fuzzy Emad-Falih Transform: General Formula of Solution Sets for Fuzzy Systems

In this section, fuzzy Emad-Falih transform technique used for solving the following system:

$$\begin{cases} \mathfrak{h}_1'(\mathfrak{t}) = \mathfrak{w}_1\left(\mathfrak{t}, \mathfrak{h}_1(\mathfrak{t}), \mathfrak{h}_2(\mathfrak{t}), \dots, \mathfrak{h}_n(\mathfrak{t})\right) \\ \mathfrak{h}_2'(\mathfrak{t}) = \mathfrak{w}_2\left(\mathfrak{t}, \mathfrak{h}_1(\mathfrak{t}), \mathfrak{h}_2(\mathfrak{t}), \dots, \mathfrak{h}_n(\mathfrak{t})\right) \\ \vdots \\ \mathfrak{h}_n'(\mathfrak{t}) = \mathfrak{w}_n\left(\mathfrak{t}, \mathfrak{h}_1(\mathfrak{t}), \mathfrak{h}_2(\mathfrak{t}), \dots, \mathfrak{h}_n(\mathfrak{t})\right) \end{cases} \quad (1)$$

Where,

$$\begin{cases} \mathfrak{h}_1(\mathfrak{t}) = \left(\underline{\mathfrak{h}_1}(\mathfrak{t},\beta), \overline{\mathfrak{h}_1}(\mathfrak{t},\beta)\right);\ \mathfrak{t} \geq 0, 0 \leq \beta \leq 1 \\ \mathfrak{h}_2(\mathfrak{t}) = \left(\underline{\mathfrak{h}_2}(\mathfrak{t},\beta), \overline{\mathfrak{h}_2}(\mathfrak{t},\beta)\right);\ \mathfrak{t} \geq 0, 0 \leq \beta \leq 1 \\ \vdots \\ \mathfrak{h}_n(\mathfrak{t}) = \left(\underline{\mathfrak{h}_n}(\mathfrak{t},\beta), \overline{\mathfrak{h}_n}(\mathfrak{t},\beta)\right);\ \mathfrak{t} \geq 0, 0 \leq \beta \leq 1, \end{cases}$$

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and $\omega_1(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))$, $\omega_2(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))$, …, $\omega_n(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))$ are linear fuzzy value functions.

Apply fuzzy Emad-Falih transform, to both side of (1) with Theorem (2.3):

$$\begin{cases} \widehat{\mathbb{EF}}[\omega_1(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))] = \begin{cases} r^2\widehat{\mathbb{EF}}[ɧ_1(ƫ)] \ominus \frac{1}{r} ɧ_1(0) & \text{if } ɧ_1 \text{ be (i)} - \text{differentiable} \\ \widehat{\mathbb{EF}}[ɧ_1(ƫ)] = -\frac{1}{r} ɧ_2(0) \ominus -r^2\widehat{\mathbb{EF}}[ɧ_1(ƫ)] & \text{if } ɧ_1 \text{ be (ii)} - \text{differentiable} \end{cases} \\ \widehat{\mathbb{EF}}[\omega_2(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))] = \begin{cases} r^2\widehat{\mathbb{EF}}[ɧ_2(ƫ)] \ominus \frac{1}{r} ɧ_2(0) & \text{if } ɧ_2 \text{ be (i)} - \text{differentiable} \\ \widehat{\mathbb{EF}}[ɧ_2(ƫ)] = -\frac{1}{r} ɧ_2(0) \ominus -r^2\widehat{\mathbb{EF}}[ɧ_2(ƫ)] & \text{if } ɧ_2 \text{ be (ii)} - \text{differentiable} \end{cases} \\ \vdots \\ \widehat{\mathbb{EF}}[\omega_n(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))] = \begin{cases} r^2\widehat{\mathbb{EF}}[ɧ_n(ƫ)] \ominus \frac{1}{r} ɧ_n(0) & \text{if } ɧ_n \text{ be (i)} - \text{differentiable} \\ \widehat{\mathbb{EF}}[ɧ_n(ƫ)] = -\frac{1}{r} ɧ_n(0) \ominus -r^2\widehat{\mathbb{EF}}[ɧ_n(ƫ)] & \text{if } ɧ_n \text{ be (ii)} - \text{differentiable} \end{cases} \end{cases}$$

Depending on the formula of the functions

$\omega_1(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))$, $\omega_2(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))$ and $\omega_n(ƫ, ɧ_1(ƫ), ɧ_2(ƫ), \dots, ɧ_n(ƫ))$ the right-hand side can be assumed with the following functions:

$$\begin{cases} \mathbb{EF}\left[\underline{ɧ_1}(ƫ, ϐ)\right] = \mathbb{P}_1(s, ϐ) \\ \mathbb{EF}\left[\overline{ɧ_1}(ƫ, ϐ)\right] = \mathbb{N}_1(s, ϐ) \\ \mathbb{EF}\left[\underline{ɧ_2}(ƫ, ϐ)\right] = \mathbb{P}_2(s, ϐ) \\ \mathbb{EF}\left[\overline{ɧ_2}(ƫ, ϐ)\right] = \mathbb{N}_2(s, ϐ) \\ \vdots \\ \mathbb{EF}\left[\underline{ɧ_n}(ƫ, ϐ)\right] = \mathbb{P}_n(s, ϐ) \\ \mathbb{EF}\left[\overline{ɧ_n}(ƫ, ϐ)\right] = \mathbb{N}_n(s, ϐ) \end{cases}$$

The solution set of system (1) is obtained by taking the inverse Emad-Falih transform for the previous system.

$$\begin{cases} \left[\underline{ɧ_1}(ƫ, ϐ)\right] = \mathbb{EF}^{-1}[\mathbb{P}_1(s, ϐ)] \\ \left[\overline{ɧ_1}(ƫ, ϐ)\right] = \mathbb{EF}^{-1}[\mathbb{N}_1(s, ϐ)] \\ \left[\underline{ɧ_2}(ƫ, ϐ)\right] = \mathbb{EF}^{-1}[\mathbb{P}_2(s, ϐ)] \\ \left[\overline{ɧ_2}(ƫ, ϐ)\right] = \mathbb{EF}^{-1}[\mathbb{N}_2(s, ϐ)] \\ \vdots \\ \left[\underline{ɧ_n}(ƫ, ϐ)\right] = \mathbb{EF}^{-1}[\mathbb{P}_n(s, ϐ)] \\ \left[\overline{ɧ_n}(ƫ, ϐ)\right] = \mathbb{EF}^{-1}[\mathbb{N}_n(s, ϐ)] \end{cases}$$

4. Application:

To provide a rough estimate of the costs of three operations in a public or private organization without an appointment, which helps investors predict earnings and expenses ahead of time and contributes to the success of investment projects. If we take into account the starting cost of each action in the following system.

$ɧ'(ʧ) = ꭒ\left(ʧ, ɧ(ʧ), Ꝏ(ʧ), ƃ(ʧ)\right) = ɧ(ʧ) + Ꝏ(ʧ) + ƃ(ʧ).$

$Ꝏ'(ʧ) = Ꮭ\left(ʧ, ɧ(ʧ), Ꝏ(ʧ), ƃ(ʧ)\right) = -Ꝏ(ʧ) - ƃ(ʧ).$

$ƃ'(ʧ) = ɷ\left(ʧ, ɧ(ʧ), Ꝏ(ʧ), ƃ(ʧ)\right) = 2ɧ(ʧ) + Ꝏ(ʧ) + ƃ(ʧ).$

Under initial fuzzy conditions:

$ɧ(0, ϐ) = (ϐ, 2 - ϐ), Ꝏ(0, ϐ) = (ϐ - 1, 1 - ϐ), ƃ(0, ϐ) = (ϐ - 3, -2ϐ).$

Apply general formula in sections 3 with Theorem (2.3) such as following:

Case (1): If $ɧ(ʧ), ƃ(ʧ)$ and $Ꝏ(ʧ)$ are (i)-differentiable, then

$$\begin{cases} \mathscr{r}^2\widehat{\mathbb{EF}}[ɧ(ʧ)] \ominus \frac{1}{\mathscr{r}}ɧ(0) = \widehat{\mathbb{EF}}[ɧ(ʧ) + Ꝏ(ʧ) + ƃ(ʧ)], \\ \mathscr{r}^2\widehat{\mathbb{EF}}[Ꝏ(ʧ)] \ominus \frac{1}{\mathscr{r}}Ꝏ(0) = \widehat{\mathbb{EF}}[-Ꝏ(ʧ) - ƃ(ʧ)]. \\ \mathscr{r}^2\widehat{\mathbb{EF}}[ƃ(ʧ)] \ominus \frac{1}{\mathscr{r}}ƃ(0) = \widehat{\mathbb{EF}}[2ɧ(ʧ) + Ꝏ(ʧ) + ƃ(ʧ)]. \end{cases}$$

After substitution initial condition.

$$\begin{cases} (\mathscr{r}^2 - 1)\mathbb{EF}\left[\underline{ɧ}(ʧ, ϐ)\right] - \mathbb{EF}[\underline{Ꝏ}(ʧ, ϐ)] - \mathbb{EF}[\underline{ƃ}(ʧ, ϐ)] = \frac{ϐ}{\mathscr{r}^3}, \\ (\mathscr{r}^2 - 1)\mathbb{EF}[\overline{ɧ}(ʧ, ϐ)] - \mathbb{EF}[\overline{Ꝏ}(ʧ, ϐ)] - \mathbb{EF}[\overline{ƃ}(ʧ, ϐ)] = \frac{2 - ϐ}{\mathscr{r}^3}, \\ \mathscr{r}^2 \mathrm{A}[\underline{Ꝏ}(ʧ, ϐ)] + \mathbb{EF}[\overline{Ꝏ}(ʧ, ϐ)] + \mathbb{EF}[\overline{ƃ}(ʧ, ϐ)] = \frac{ϐ - 1}{\mathscr{r}^3}, \\ \mathscr{r}^2 \mathrm{A}[\overline{Ꝏ}(ʧ, ϐ)] + \mathbb{EF}[\underline{Ꝏ}(ʧ, ϐ)] + \mathbb{EF}[\underline{ƃ}(ʧ, ϐ)] = \frac{1 - ϐ}{\mathscr{r}^3}, \\ (\mathscr{r}^2 - 1)\mathbb{EF}[\underline{ƃ}(ʧ, ϐ)] - 2\mathbb{EF}\left[\underline{ɧ}(ʧ, ϐ)\right] - \mathbb{EF}[\underline{Ꝏ}(ʧ, ϐ)] = \frac{ϐ - 3}{s}, \\ (\mathscr{r}^2 - 1)\mathbb{EF}[\overline{ƃ}(ʧ, ϐ)] - 2\mathbb{EF}[\overline{ɧ}(ʧ, ϐ)] - \mathbb{EF}[\overline{Ꝏ}(ʧ, ϐ)] = \frac{-2ϐ}{\mathscr{r}^3}. \end{cases}$$

With simple calculation and using the inverse Emad-Falih transform obtained the solution of case (1)

$$\underline{ɧ}(ʧ, ϐ) = ϐ\left(\frac{-1}{6} + \frac{1}{6}\,e^{-ʧ} - \frac{1}{6}\,e^{2ʧ} + \frac{7}{6}\,e^{3ʧ}\right) + \frac{1}{6} + \frac{5}{6}\,e^{-ʧ} + \frac{1}{6}\,e^{2ʧ} - \frac{7}{6}\,e^{3ʧ}.$$

$$\overline{ɧ}(ʧ, ϐ) = ϐ\left(\frac{1}{6} + \frac{1}{12}\,e^{-ʧ} - \frac{5}{6}\,e^{2ʧ} - \frac{5}{12}\,e^{3ʧ}\right) - \frac{1}{6} + \frac{5}{6}\,e^{-ʧ} + \frac{1}{6}\,e^{2ʧ} + \frac{7}{6}\,e^{3ʧ}.$$

$$\underline{Ꝏ}(ʧ, ϐ) = ϐ\left(\frac{17}{36} + \frac{1}{6}\,ʧ - \frac{1}{3}\,e^{-ʧ} + \frac{1}{12}\,e^{2ʧ} + \frac{7}{9}\,e^{3ʧ}\right) + \frac{55}{36} - \frac{1}{6}ʧ - \frac{5}{3}\,e^{-ʧ} - \frac{1}{12}\,e^{2ʧ} - \frac{7}{9}\,e^{3ʧ}.$$

$$\overline{Ꝏ}(ʧ, ϐ) = ϐ\left(\frac{1}{36} - \frac{1}{6}\,ʧ - \frac{1}{3}\,e^{-ʧ} + \frac{1}{12}\,e^{2ʧ} - \frac{7}{9}\,e^{3ʧ}\right) + \frac{71}{36} + \frac{1}{6}ʧ - \frac{5}{3}\,e^{-ʧ} - \frac{1}{12}\,e^{2ʧ} + \frac{7}{9}\,e^{3ʧ}.$$

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$$\underline{\text{ƅ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{-11}{36} - \frac{1}{6}\,\text{ʧ} - \frac{1}{4}\,e^{2\text{ʧ}} + \frac{14}{9}e^{3\text{ʧ}}\right)\frac{-61}{36} + \frac{1}{6}\,\text{ʧ} + \frac{1}{4}\,e^{2\text{ʧ}} - \frac{14}{9}e^{3\text{ʧ}}.$$

$$\overline{\text{ƅ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{-7}{36} + \frac{1}{6}\,\text{ʧ} - \frac{1}{4}\,e^{2\text{ʧ}} - \frac{14}{9}e^{3\text{ʧ}}\right)\frac{-65}{36} - \frac{1}{6}\,\text{ʧ} + \frac{1}{4}\,e^{2\text{ʧ}} + \frac{14}{9}e^{3\text{ʧ}}.$$

Case (2): If $\text{ɧ}(\text{ʧ})$ and $\text{ƅ}(\text{ʧ})$ are (i)-differentiable but $\text{Ծ}(\text{ʧ})$ is (ii)-differentiable, then similar with case (1), by taking Emad-Falih transform and substitution, initial condition, yields:

$$\begin{cases}
(r^2-1)\mathbb{EF}\left[\underline{\text{ɧ}}(\text{ʧ},\text{ϐ})\right] - \mathbb{EF}\left[\underline{\text{Ծ}}(\text{ʧ},\text{ϐ})\right] - \mathbb{EF}\left[\underline{\text{ƅ}}(\text{ʧ},\text{ϐ})\right] = \dfrac{\text{ϐ}}{r^3},\\
(r^2-1)\mathbb{EF}\left[\overline{\text{ɧ}}(\text{ʧ},\text{ϐ})\right] - \mathbb{EF}\left[\overline{\text{Ծ}}(\text{ʧ},\text{ϐ})\right] - \mathbb{EF}\left[\overline{\text{ƅ}}(\text{ʧ},\text{ϐ})\right] = \dfrac{2-\text{ϐ}}{r^3},\\
(r^2+1)\mathbb{EF}\left[\underline{\text{Ծ}}(\text{ʧ},\text{ϐ})\right] + \mathbb{EF}\left[\left[\underline{\text{ƅ}}(\text{ʧ},\text{ϐ})\right] = \dfrac{\text{ϐ}-1}{r^3},\\
(r^2+1)\mathbb{EF}\left[\overline{\text{Ծ}}(\text{ʧ},\text{ϐ})\right] + \mathbb{EF}\left[\overline{\text{ƅ}}\,(\text{ʧ},\text{ϐ})\right] = \dfrac{1-\text{ϐ}}{r^3},\\
(r^2-1)\mathbb{EF}\left[\underline{\text{ƅ}}(\text{ʧ},\text{ϐ})\right] - 2\mathbb{EF}\left[\underline{\text{ɧ}}(\text{ʧ},\text{ϐ})\right] - \mathbb{EF}\left[\underline{\text{Ծ}}(\text{ʧ},\text{ϐ})\right] = \dfrac{\text{ϐ}-3}{r^3},\\
(r^2-1)\mathbb{EF}\left[\overline{\text{ƅ}}(\text{ʧ},\text{ϐ})\right] - 2\mathbb{EF}\left[\overline{\text{ɧ}}(\text{ʧ},\text{ϐ})\right] - \mathbb{EF}\left[\overline{\text{Ծ}}(\text{ʧ},\text{ϐ})\right] = \dfrac{-2\text{ϐ}}{r^3}.
\end{cases}$$

With simple calculation and using the inverse Emad-Falih transform obtained the solution of case (2)

$$\underline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(-\frac{1}{3}\,e^{-\text{ʧ}} + \frac{4}{3}\,e^{2\text{ʧ}}\right) - \frac{4}{3}\,e^{-\text{ʧ}} + \frac{4}{3}\,e^{2\text{ʧ}}.$$

$$\overline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{2}{3}\,e^{-\text{ʧ}} - \frac{5}{3}\,e^{2\text{ʧ}}\right) + \frac{1}{3}\,e^{-\text{ʧ}} + \frac{5}{3}\,e^{2\text{ʧ}}.$$

$$\underline{\text{Ծ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(1 + \frac{2}{3}e^{-\text{ʧ}} - \frac{2}{3}\,e^{2\text{ʧ}}\right) + 1 - \frac{8}{3}e^{-\text{ʧ}} + \frac{2}{3}\,e^{2\text{ʧ}}.$$

$$\overline{\text{Ծ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(-\frac{1}{2} - \frac{4}{3}e^{-\text{ʧ}} + \frac{5}{6}\,e^{2\text{ʧ}}\right) + \frac{5}{2} - \frac{2}{3}e^{-\text{ʧ}} - \frac{5}{6}\,e^{2\text{ʧ}}.$$

$$\underline{\text{ƅ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(-1 - 2e^{2\text{ʧ}}\right) + 1 - 2e^{2\text{ʧ}}.$$

$$\overline{\text{ƅ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{1}{2} - \frac{5}{2}e^{2\text{ʧ}}\right) - \frac{5}{2} + \frac{5}{2}e^{2\text{ʧ}}.$$

Case (3): If $\text{ɧ}(\text{ʧ})$ and $\text{Ծ}(\text{ʧ})$ are (i)-differentiable but $\text{ƅ}(\text{ʧ})$ is (ii)-differentiable, then

$$\begin{cases}
r^2\widehat{\mathbb{EF}}[\text{ɧ}(\text{ʧ})] \ominus \dfrac{1}{r}\text{ɧ}(0) = \widehat{\mathbb{EF}}\,[\text{ɧ}(\text{ʧ}) + \text{Ծ}(\text{ʧ}) + \text{ƅ}(\text{ʧ})],\\
r^2\widehat{\mathbb{EF}}[\text{Ծ}(\text{ʧ})] \ominus \dfrac{1}{r}\text{Ծ}(0) = \widehat{\mathbb{EF}}[-\text{Ծ}(\text{ʧ}) - \text{ƅ}(\text{ʧ})],\\
-\dfrac{1}{r}\text{ƅ}(0) \ominus -r^2\widehat{\mathbb{EF}}[\text{ƅ}(\text{ʧ})] = \widehat{\mathbb{EF}}[2\text{ɧ}(\text{ʧ}) + \text{Ծ}(\text{ʧ}) + \text{ƅ}(\text{ʧ})].
\end{cases}$$

With simple calculation and using the inverse Emad-Falih transform obtained the solution of case (3)

$$\underline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{1}{2}e^{-\text{ʧ}} - \frac{1}{2}e^{2\text{ʧ}} + e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{6}{7}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right) + \frac{5}{6}e^{-\text{ʧ}} + \frac{1}{6}e^{2\text{ʧ}} - e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) - \frac{6}{7}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

$$\overline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{1}{6}e^{-\text{ʧ}} - \frac{1}{6}e^{2\text{ʧ}} - e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{6}{\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right) + \frac{5}{6}e^{-\text{ʧ}} + \frac{1}{6}e^{2\text{ʧ}} + e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{6}{\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

$$\underline{\text{Ծ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(-1 - \frac{1}{3}e^{-\text{ʧ}} + \frac{1}{12}e^{2\text{ʧ}} + \frac{9}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{11}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right) + 3 - \frac{5}{3}e^{-\text{ʧ}} - \frac{1}{12}e^{2\text{ʧ}} - \frac{9}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) - \frac{11}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

$$\overline{\text{Ծ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{3}{2} - \frac{1}{3}e^{-\text{ʧ}} + \frac{1}{12}e^{2\text{ʧ}} - \frac{9}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) - \frac{11}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right) + \frac{1}{2} - \frac{5}{3}e^{-\text{ʧ}} - \frac{1}{12}e^{2\text{ʧ}} + \frac{9}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{11}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

$$\underline{\text{ƅ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(1 - \frac{1}{4}e^{2\text{ʧ}} + \frac{1}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) - \frac{37}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right) - 6 + \frac{7}{4}e^{2\text{ʧ}} + \frac{5}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{\sqrt{7}}{4}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

$$\overline{\text{ƅ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{-3}{2} - \frac{1}{4}e^{2\text{ʧ}} - \frac{1}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) + \frac{37}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right) - \frac{1}{2} + \frac{1}{4}e^{2\text{ʧ}} + \frac{1}{4}e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right) - \frac{37}{4\sqrt{7}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

Case (4): If $\text{Ծ}(\text{ʧ})$ and $\text{ƅ}(\text{ʧ})$ are (i)-differentiable but $\text{ɧ}(\text{ʧ})$ is (ii)-differentiable, then

$$\begin{cases} -\frac{1}{r}\text{ɧ}(0) \ominus -r^{2}\widehat{\mathbb{EF}}[\text{ɧ}(\text{ʧ})] = \widehat{\mathbb{EF}}\,[\text{ɧ}(\text{ʧ}) + \text{Ծ}(\text{ʧ}) + \text{ƅ}(\text{ʧ})], \\ r^{2}\widehat{\mathbb{EF}}[\text{Ծ}(\text{ʧ})] \ominus \frac{1}{r}\text{Ծ}(0) = \widehat{\mathbb{EF}}[-\text{Ծ}(\text{ʧ}) - \text{ƅ}(\text{ʧ})], \\ r^{2}\widehat{\mathbb{EF}}[\text{ƅ}(\text{ʧ})] \ominus \frac{1}{r}\text{ƅ}(0) = \widehat{\mathbb{EF}}[2\text{ɧ}(\text{ʧ}) + \text{Ծ}(\text{ʧ}) + \text{ƅ}(\text{ʧ})]. \end{cases}$$

With simple calculation and using the inverse Emad-Falih transform obtained the solution of case (4)

$$\underline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{1}{6}e^{-\text{ʧ}} - \frac{1}{6}e^{2\text{ʧ}} + e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{17}\text{ʧ}}{2}\right) + \frac{2}{\sqrt{17}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{17}\text{ʧ}}{2}\right)\right) + \frac{5}{6}e^{-\text{ʧ}} + \frac{1}{6}e^{2\text{ʧ}} - e^{\frac{\text{ʧ}}{2}}\cos\left(\frac{\sqrt{17}\text{ʧ}}{2}\right) - \frac{2}{\sqrt{17}}e^{\frac{\text{ʧ}}{2}}\sin\left(\frac{\sqrt{17}\text{ʧ}}{2}\right).$$

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$$\overline{ɧ}(ʧ,ϐ)=ϐ\left(\frac{1}{6}e^{-ʧ}-\frac{1}{6}e^{2ʧ}-e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)-\frac{2}{\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right)\right)+\frac{5}{6}e^{-ʧ}+\frac{1}{6}e^{2ʧ}+e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)+\frac{2}{\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right).$$

$$\underline{Ծ}(ʧ,ϐ)=ϐ\left(\frac{1}{8}-\frac{1}{3}e^{-ʧ}+\frac{1}{12}e^{2ʧ}+\frac{9}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)+\frac{31}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right)\right)+\frac{15}{8}-\frac{5}{3}e^{-ʧ}-\frac{1}{12}e^{2ʧ}-\frac{9}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)-\frac{31}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right).$$

$$\overline{Ծ}(ʧ,ϐ)=ϐ\left(\frac{3}{8}-\frac{1}{3}e^{-ʧ}+\frac{1}{12}e^{2ʧ}-\frac{9}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)-\frac{31}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right)\right)+\frac{13}{8}-\frac{5}{3}e^{-ʧ}-\frac{1}{12}e^{2ʧ}+\frac{9}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)+\frac{31}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right).$$

$$\underline{ƅ}(ʧ,ϐ)=ϐ\left(-\frac{1}{8}-\frac{1}{4}e^{2ʧ}+\frac{11}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)+\frac{61}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right)\right)-\frac{15}{8}+\frac{1}{4}e^{2ʧ}-\frac{11}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)-\frac{61}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right).$$

$$\overline{ƅ}(ʧ,ϐ)=ϐ\left(-\frac{3}{8}-\frac{1}{4}e^{2ʧ}-\frac{11}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)-\frac{61}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right)\right)-\frac{13}{8}+\frac{1}{4}e^{2ʧ}+\frac{11}{8}e^{\frac{ʧ}{2}}\cos\left(\frac{\sqrt{17}ʧ}{2}\right)+\frac{61}{8\sqrt{17}}e^{\frac{ʧ}{2}}\sin\left(\frac{\sqrt{17}ʧ}{2}\right).$$

Case (5): If $ɧ(ʧ)$, $ƅ(ʧ)$ and $Ծ(ʧ)$ are (ii)-differentiable, then

$$\underline{ɧ}(ʧ,ϐ)=ϐ\left(-\frac{1}{6}+\frac{1}{6}e^{-ʧ}-\frac{1}{6}e^{2ʧ}+\frac{7}{6}e^{-3ʧ}\right)+\frac{1}{6}+\frac{5}{6}e^{-ʧ}+\frac{1}{2}e^{2ʧ}-\frac{7}{6}e^{-3ʧ}.$$

$$\overline{ɧ}(ʧ,ϐ)=ϐ\left(\frac{1}{6}+\frac{1}{6}e^{-ʧ}-\frac{1}{6}e^{2ʧ}-\frac{7}{6}e^{-3ʧ}\right)-\frac{1}{6}+\frac{5}{6}e^{-ʧ}+\frac{1}{2}e^{2ʧ}+\frac{7}{6}e^{-3ʧ}.$$

$$\underline{Ծ}(ʧ,ϐ)=ϐ\left(\frac{17}{36}-\frac{1}{6}ʧ-\frac{1}{3}e^{-ʧ}+\frac{1}{12}e^{2ʧ}+\frac{7}{9}e^{-3ʧ}\right)+\frac{55}{36}+\frac{1}{6}ʧ-\frac{5}{3}e^{-ʧ}-\frac{1}{12}e^{2ʧ}-\frac{7}{9}e^{-3ʧ}.$$

$$\overline{Ծ}(ʧ,ϐ)=ϐ\left(\frac{1}{36}+\frac{1}{6}ʧ-\frac{1}{3}e^{-ʧ}+\frac{1}{12}e^{2ʧ}-\frac{7}{9}e^{-3ʧ}\right)+\frac{71}{36}-\frac{1}{6}ʧ-\frac{5}{3}e^{-ʧ}-\frac{1}{12}e^{2ʧ}+\frac{7}{9}e^{-3ʧ}.$$

$$\underline{ƅ}(ʧ,ϐ)=ϐ\left(\frac{-11}{36}+\frac{1}{6}ʧ-\frac{1}{4}e^{2ʧ}+\frac{14}{9}e^{-3ʧ}\right)+\frac{-16}{36}-\frac{1}{6}ʧ+\frac{1}{4}e^{2ʧ}-\frac{14}{9}e^{-3ʧ}.$$

$$\overline{ƅ}(ʧ,ϐ)=ϐ\left(\frac{-7}{36}-\frac{1}{6}ʧ-\frac{1}{4}e^{2ʧ}-\frac{14}{9}e^{-3ʧ}\right)+\frac{-65}{36}+\frac{1}{6}ʧ+\frac{1}{4}e^{2ʧ}+\frac{14}{9}e^{-3ʧ}.$$

Case (6): If $ɧ(ʧ)$ and $ƅ(ʧ)$ are (ii)-differentiable but $Ծ(ʧ)$ is (i)-differentiable, then

$$\underline{ɧ}(ʧ,ϐ)=ϐ\left(\frac{1}{6}e^{-ʧ}-\frac{1}{2}e^{ʧ}+\frac{3}{2}e^{-2ʧ}-\frac{1}{6}e^{2ʧ}\right)-\frac{1}{6}e^{-ʧ}+\frac{1}{2}e^{ʧ}+\frac{3}{2}e^{-2ʧ}+\frac{7}{6}e^{2ʧ}.$$

$$\overline{ɧ}(ʧ,ϐ)=ϐ\left(\frac{1}{6}e^{-ʧ}+\frac{1}{2}e^{ʧ}-\frac{3}{2}e^{-2ʧ}-\frac{1}{6}e^{2ʧ}\right)+\frac{13}{6}e^{-ʧ}-\frac{11}{2}e^{ʧ}+\frac{5}{6}e^{-2ʧ}+\frac{5}{6}e^{2ʧ}.$$

$$\underline{\text{Ʊ}}(\text{ʧ},\beta)=\beta\left(1-\frac{1}{3}\,e^{-\text{ʧ}}+e^{\text{ʧ}}-\frac{3}{4}\,e^{-2\text{ʧ}}+\frac{1}{12}\,e^{2\text{ʧ}}\right)+1-\frac{5}{3}\,e^{-\text{ʧ}}-e^{\text{ʧ}}+\frac{3}{4}\,e^{-2\text{ʧ}}-\frac{1}{12}\,e^{2\text{ʧ}}.$$

$$\overline{\text{Ʊ}}(\text{ʧ},\beta)=\beta\left(-\frac{1}{2}-\frac{1}{3}\,e^{-\text{ʧ}}-e^{\text{ʧ}}+\frac{3}{4}\,e^{-2\text{ʧ}}+\frac{1}{12}\,e^{2\text{ʧ}}\right)+\frac{5}{2}-\frac{5}{3}\,e^{-\text{ʧ}}+e^{\text{ʧ}}-\frac{3}{4}\,e^{-2\text{ʧ}}-\frac{1}{12}\,e^{2\text{ʧ}}.$$

$$\underline{\text{ƀ}}(\text{ʧ},\beta)=\beta\left(-1+\frac{9}{4}\,e^{-2\text{ʧ}}-\frac{1}{4}\,e^{2\text{ʧ}}\right)+1-\frac{9}{4}\,e^{-2\text{ʧ}}+\frac{1}{4}\,e^{2\text{ʧ}}.$$

$$\overline{\text{ƀ}}(\text{ʧ},\beta)=\beta\left(\frac{1}{2}-\frac{9}{4}\,e^{-2\text{ʧ}}-\frac{1}{4}\,e^{2\text{ʧ}}\right)-\frac{5}{2}+\frac{9}{4}\,e^{-2\text{ʧ}}+\frac{1}{4}\,e^{2\text{ʧ}}.$$

Case (7): If $\text{ɧ}(\text{ʧ})$ and $\text{Ʊ}(\text{ʧ})$ are (ii)-differentiable but $\text{ƀ}(\text{ʧ})$ is (i)-differentiable, then

$$\underline{\text{ɧ}}(\text{ʧ},\beta)=\beta\left(e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{6}{\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{1}{6}\,e^{-\text{ʧ}}-\frac{1}{6}\,e^{2\text{ʧ}}\right)+\frac{1}{2}e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)$$
$$+\frac{3}{2\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{11}{6}\,e^{-\text{ʧ}}+\frac{2}{3}\,e^{2\text{ʧ}}.$$

$$\overline{\text{ɧ}}(\text{ʧ},\beta)=\beta\left(-\,e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{6}{\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{1}{6}\,e^{-\text{ʧ}}-\frac{1}{6}\,e^{2\text{ʧ}}\right)+e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)$$
$$-\frac{6}{\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{5}{6}\,e^{-\text{ʧ}}+\frac{1}{6}\,e^{2\text{ʧ}}.$$

$$\underline{\text{Ʊ}}(\text{ʧ},\beta)=\beta\left(-1+\frac{9}{4}\,e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{11}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{1}{3}\,e^{-\text{ʧ}}+\frac{1}{12}\,e^{2\text{ʧ}}\right)+3$$
$$-\frac{9}{4}e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{11}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{5}{3}\,e^{-\text{ʧ}}-\frac{1}{12}\,e^{2\text{ʧ}}.$$

$$\overline{\text{Ʊ}}(\text{ʧ},\beta)=\beta\left(\frac{3}{2}-\frac{9}{4}\,e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{11}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{1}{3}\,e^{-\text{ʧ}}+\frac{1}{12}\,e^{2\text{ʧ}}\right)+\frac{1}{2}+\frac{9}{4}e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)$$
$$-\frac{11}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{5}{3}\,e^{-\text{ʧ}}-\frac{1}{12}\,e^{2\text{ʧ}}.$$

$$\underline{\text{ƀ}}(\text{ʧ},\beta)=\beta\left(1-\frac{1}{4}\,e^{2\text{ʧ}}+\frac{1}{4}\,e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)+\frac{37}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right)-3+\frac{1}{4}\,e^{2\text{ʧ}}-\frac{1}{4}e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)$$
$$-\frac{37}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

$$\overline{\text{ƀ}}(\text{ʧ},\beta)=\beta\left(-\frac{3}{2}-\frac{1}{4}\,e^{2\text{ʧ}}-\frac{1}{4}\,e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)-\frac{37}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)\right)-\frac{1}{2}+\frac{1}{4}\,e^{2\text{ʧ}}+\frac{1}{4}e^{\frac{-\text{ʧ}}{2}}\cos\left(\frac{\sqrt{7}\text{ʧ}}{2}\right)$$
$$+\frac{37}{4\sqrt{7}}e^{\frac{-\text{ʧ}}{2}}\sin\left(\frac{\sqrt{7}\text{ʧ}}{2}\right).$$

Case (8): Finally, if $\text{Ʊ}(\text{ʧ})$ and $\text{ƀ}(\text{ʧ})$ are (ii)-differentiable but $\text{ɧ}(\text{ʧ})$ is (i)-differentiable, then solution set for the last case takes the following form :

$$\underline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{9}{2} - \frac{10}{3}\mathrm{e}^{-\text{ʧ}} - \frac{1}{6}\mathrm{e}^{2\text{ʧ}}\right) - \frac{9}{2} + \frac{13}{3}\mathrm{e}^{-\text{ʧ}} + \frac{1}{6}\mathrm{e}^{2\text{ʧ}}.$$

$$\overline{\text{ɧ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(-\frac{9}{2} + \frac{11}{3}\mathrm{e}^{-\text{ʧ}} - \frac{1}{6}\mathrm{e}^{2\text{ʧ}}\right) + \frac{9}{2} - \frac{8}{3}\mathrm{e}^{-\text{ʧ}} + \frac{1}{6}\mathrm{e}^{2\text{ʧ}}.$$

$$\underline{\text{Ծ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{-23}{4} + \frac{9}{2}\text{ʧ} + \frac{20}{3}\,\mathrm{e}^{-\text{ʧ}} + \frac{1}{12}\mathrm{e}^{2\text{ʧ}}\right) + \frac{31}{4} - \frac{9}{2}\text{ʧ} - \frac{36}{3}\,\mathrm{e}^{-\text{ʧ}} - \frac{1}{12}\mathrm{e}^{2\text{ʧ}}.$$

$$\overline{\text{Ծ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{25}{4} - \frac{9}{2}\text{ʧ} - \frac{22}{3}\,\mathrm{e}^{-\text{ʧ}} + \frac{1}{12}\mathrm{e}^{2\text{ʧ}}\right) - \frac{17}{4} + \frac{9}{2}\text{ʧ} + \frac{16}{3}\,\mathrm{e}^{-\text{ʧ}} - \frac{1}{12}\mathrm{e}^{2\text{ʧ}}.$$

$$\underline{\text{ƀ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{5}{4} - \frac{9}{2}\text{ʧ} - \frac{1}{4}\,\mathrm{e}^{2\text{ʧ}}\right) - \frac{13}{4} + \frac{9}{2}\text{ʧ} + \frac{1}{4}\mathrm{e}^{2\text{ʧ}}.$$

$$\overline{\text{ƀ}}(\text{ʧ},\text{ϐ}) = \text{ϐ}\left(\frac{-7}{4} + \frac{9}{2}\text{ʧ} - \frac{1}{4}\,\mathrm{e}^{2\text{ʧ}}\right) - \frac{1}{4} - \frac{9}{2}\text{ʧ} + \frac{1}{4}\mathrm{e}^{2\text{ʧ}}.$$

Conclusion:

A system of n-dimensional fuzzy linear first order differential equations is solved using the fuzzy Emad-Falih transform. We provide a real-world example to demonstrate the effectiveness and caliber of the approach.

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