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STUDY OF GENERALIZATION'S OF MAXIMAL IDEAL & MAXIMAL SUBMODULE ((GENERAL-MAXIMAL IDEAL & TWO-PURE MAXIMAL SUBMODULE))

Annotation:

Maximal ideal of R⇔ $\frac{R}{GEM}$ is n-regular ring.(R left(right) community ring)
*And we define the concept Two-pure Maximal submodule of R-module (TPM–submodule), where a submodule TPM of R-Module M is called Two-pure Maximal submodule of R if there exist a submodule d ⊆ M and TPM ⊆ d ⊆ M such that d is 2-pure submodule of R . TPM- submodule & GEM this concept *generalization* of Maximal ideals & TPM- submodule this concept *generalization* of Maximal submodule, Many features and properties of (TPM & GEM) are given.*

Keywords: *TPM (2-pure maximal R-module), GEM (General-Maximal ideal of R), íR (an ideal of R), Max (Maximal ideal of R).*

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1. Introductíon

Burton in [5] development of Mαximal ideals, where íR is called Mαx ⇔ $\frac{R}{íR}$ is simple rings also K is called Mαximal submodule of R-module M ⇔ $\frac{M}{K}$ is simple R-module . Furthermore, Goodyear K.R. [7] introduced the concept of semi-Mαx & semi-Mαximal submodule of M *generalization* of Mαximal ideal and Mαximal submodule of M where íR is called semi-Mαx ⇔ $\frac{R}{íR}$ is a semi simple ring. Shwkaea M.R in [8], gives two led called weak-Mαximal ideal & weak-Mαximal submodule of M , where íR is called a weak-Max ⇔ $\frac{R}{íR}$ is regular ríng .

Also Raid D.Mahamood &Mohammed [1] introduced the concept of regular where a ring R is called regular if every ideal íR is pure in R . Liang, Z. and Gang in [2] introduced the concept of n-regular where a ring R is called n-regulαr if a ∈ aRa for all a ∈ N(R) , *every regular is n-regular* in this paper introduced another *generalizations* of Max ideal known **GEM and** & Mαximal submodule of M known **TPM** , This concept is a *generalization* of the concepts of semi & (weak)-Maximal submodule & pure- Maximal submodule. so, we gives (examine properties, characteristics, and examples of **GEM &TPM** .

2. GEM- ideal.

This section defines **General-Max ideal**(GEM) characterization and properties of **GEM** gives.

Define (2.1)

GEM is called **General** – Maximal ideal of R⇔ $\frac{R}{GEM}$ is n-regular ring.

Remarks and Examples (2 . 2)

(1) Each Max ideal is **GEM** , whilst the discuss is not success in generally, so by example 10Z is **GEM** of Z , but 10Z is not Max of Z , since $\frac{Z}{10Z} \cong Z_{10}$ is n-regular but not simple.

(2) 14Z and 15Z of Z are **GEM**.

(3) Each semi-Max Is **GEM**, but discuss is not.

(4) Each weak Max is **GEM**, But the converse is not true .

Proposition (2 . 3)

If K , G be two ideal such that K ⊆ G .Then

(1) G is **GEM** ⇔ G/K is **GEM of** R/ K .

(2) Given K is **GEM** then G is **GEM** .

proof :

(1) Take G be **GEM** then R/G is n-regular group. Since K ⊆ G, then G/K is ideal of R/K. Hence by *third - isomorphism theorem* we get R/G ≅ R/K⁄G/K which implies that R/K⁄G/K is n-regular group , Hence G/K is **GEM** of R/K .

For conversely , take G/K is **GEM** of R/K, so it is n-regular group , and again by *third isomorphism theorem* we get R/G is n-regular group , which signify G is **GEM** .

(2) Since K is **GEM** , then R/K is n-regular group .Since K ⊆ G, then G/K is ideal of R/K , R/K⁄G/K is n-regular ring thus by third isomorphism theorem we get R/G is n-regular ring G is **GEM** .

From(Proposition 2.3) we get some result :-

Lemma (2 . 4)

If G is **GEM**, then $\sqrt{G}$ is **GEM**

proof:

Since G ⊆ $\sqrt{G}$ thus that prove its clear by proposition(2.3)

lemma (2 . 5)

if $(\sqrt{K})^n$ is **GEM** , R is Noetherian ring ⟹ K is **GEM** .

proof:

Since R is Noetherian group, at that time by proposition:- $\left(\sqrt{íR}\right)^n \subseteq íR$ for any $n \in z^+$. where a ring R is Noetherian if R satisfy an climbing chain condition for ideal of R [5] .so that $\left(\sqrt{K}\right)^n \subseteq K$ so by pro(2.3) K is **GEM** .

lemma (2 . 6)

If K , G be two ideal , and K ∩ G is **GEM** ⟹ K , G are **GEM** .

proof:

Since Ꝃ ∩ Ǵ ⊆ Ꝃ and Ꝃ ∩ Ǵ ⊆ Ǵ for each ideals Ǵ therefor by proposition(2.3) prove clearly

lemma(2 . 7)

If Ꝃ , Ǵ be two íR , and Ꝃ or Ǵ are **GEM** , Then Ꝃ +Ǵ is **GEM** .

lemma(2 . 8)

If Ꝃ is an ideal , and (J(R) + Ꝃ)/Ꝃ is **GEM** , Then J(R/Ꝃ) is **GEM of** R/Ꝃ .

lemma(2 . 9) [2]

Every one sided or two sided nil ideal of Ris contained in J(R).

Lemma (2. 10)[2]

Let R be n-regular then N(R)∩ J(R)-0

Theorem (2 . 11)[2]

If N(R) is íR and R is strongly regular group ↔ R is n-regular and R/N(R) is regular .

Proposition (2 . 12)

let R is strongly regular group R, then N(R) is **GEM**

proof:

Since R is strongly regular so by [proposition 2.11] R/N(R) is regular since every regular group is n-regular then R/N(R) is n- regular hence N(R) is **GEM**.

It is clear that N(R) weak- Max ideal.

Proposition (2 . 13)

if Ꝃ is weak-Max then Ꝃ is **GEM**

proof:

take Ꝃ be weak-Max then R/Ꝃ is regular group so its n-regular ,hence N is **GEM**

Proposition (2 . 14)

N(R) be an ideal of reduce ríng R → N(R) is **GEM**.

proof:

Since R is reduce so R is ɳ-regular[2], N(R) in R by [lemma 2 . 9] N(R) □□J(R). But R is n-regular so N(R) □□J(R)=0 , [lemma 2 . 10] , So N(R)□□R(R)=N(R)=0 Since R is n-regular & R/N(R) □□R/{0} so R/N(R) is n-regular so N(R) is **GEM**.

lemma (2 . 15)

if Ra is a direct call and in R then N(R) is **GEM** for all a ∈ N(R).

proof:

take Ra is a direct call in R therefore by [proposition 2.11] R is n-regular so prove is clearly.

Proposition (2 . 16)

is **GEM** for either group R. J(R)

proof:

By [4,theorem(15-17)] Ʀ/J(Ʀ) is semi-simple ring ,since every single semi-simple group is ɍegular so Ʀ/J(Ʀ) is n-ɍegulɑr therefore J(R) is **GEM**.

Proposition (2 . 17)

If N be a principal íR with property Ʀ/N is finite group then N is **GEM**

proof:

Since N is a principal , then R/N is an *integral domain* . and it is finite hence it is a field so that Ʀ/N is ɍegular ríng so its n-ɍegulɑr then N is **GEM**

Proposition (2 . 18)

Take Ꝁ is íR then Ꝁ is **GEM** if each number one ideal of Ʀ/Ꝁ is generated by an impotency element .

proof:

by [6,theorem .1.1] Ʀ/Ꝁ is ɍegular so its n-ɍegulɑr then Ꝁ is **GEM**.

Proposition (2 . 19)

let Ꝁ is íR then Ꝁ is **GEM** if all principal íR is * directs summoned* .

proof:

by [6,theorem .1.1] clearly N is **GEM .**

Proposition (2 . 20)

let Ꝁ is íR then Ꝁ is **GEM** if all * finitely generated* íR is generated by an idempotent .

proof:

by the same way by using [6,theorem .1.1]

Proposition (2 . 21)

let Ꝁ is íR then Ꝁ is **GEM** if all element written as a*product of unit and idempotent*.

proof:

By proposition a group R is ɍegular ⇔ all elements written as a *product of a unit and idempotent* . where an element x of a group R with identity 1_R , is called a unit , if there breathe y ∈ R such that xy = 1_R .[3] so that R is ɍegular group hence Ʀ/Ꝁ is ɍegular group so its n-ɍegulɑr therefore it is **GEM** .

Proposition (2 . 22)

Let Ǵ is íR then Ǵ is **GEM** if Ʀ/Ǵ is semi-prime for each ideal.

proof:

Since R/Ǵ is semi-prime .therefore by [5,theorem. 9.6] Ʀ/Ǵ is ɍegular group so its n-ɍegulɑr hence Ǵ is **GEM**

Now we define the concept 2-pure maximal submodule

3. TⱣⱮ -submodule

This section defines Two-pure- Mɑximal submodule (TⱣⱮ,) characterization and properties of TⱣⱮ -submodule gives .

Define (2. 1)

a submodule TPM of R-Module M is called Two-pure Maximal submodule of R if there exist a submodule d ⊆ M and TPM ⊆ d ⊆ M such that d is 2-pure submodule of R

Remarks and Examples (3 . 2)

1. Each Tow- Maximal in M is TPM.

Proof:- let p is Two-Maximal in M and let N ⊆ M s.t p ⊆ N ⊆ M then M/p is 2-regulare [10] so by [9,proposition,1.1.15] M is 2-regular hence all submodule in M is 2-pure so N is 2-pure submodule then p is TPM

2. 6Z is TPM of z-module z since $z_6 \cong \frac{z}{6z}$ is 2-regular so that every submodule is 2-pure submodule
3. 25Z of z-module and 6Z of z-module z_{12} are TPM - submodule.
4. Each Max submodule of M is TPM - submodule.

Proof:- let p Max submodule of M so there exist N ⊆ M s.t p ⊆ N ⊆ M so N=M but N is pure submodule so that N is 2-pure submodule of M (every pure submodule is 2-p submodule) so that p is TPM- submodule. The conversely is not true

For examples :- Z_6 is not simple but it is 2-regular so 6z is TPM-submodule but not max submodule.

5. If p in R-module M and $\frac{M}{p}$ is simple R-module then p is TPM.

Proof:- since $\frac{M}{p}$ is simple R-module then p is then p is Max submodule of M so by above remark prove clearly .

6. If p is pure submodule →every proper submodule contain in p is TPM .

Propositon (3 . 3)

1. any submodule in regular R-module M is TPM.
2. any submodule in 2-regular R-module M is TPM.

Proof:-

1. Let p , N are submodule in M s.t p ⊆ N ⊆ M but M is regular so any submodule in M is pure and every pure in M is 2-pure hence p is TPM.
2. prove is clearly .

Proposition (3 . 4)

1. If p any submodule of simple R-module M⟹ p is TPM.
2. If p is pure submodule of R-module M and $\frac{M}{p}$ is regular regular R-module ⟹ p is TPM.
3. If p is 2-p submodule of R-module M and $\frac{M}{p}$ is 2-regular regular R-module ⟹ p is TPM.

Proof:-

1. since M is simple so that ever submodule of M is max-submodule therefore by[4, remark 3.2] p is TPM .
2. prove clearly
3. let p ,N are submodule of M such that p ⊆ N ⊆ M but $\frac{M}{p}$ is 2-r egular so $\frac{N}{p}$ is 2-p submodule since p is 2-p so that N is 2-p submodule[9] therefore p is TPM.

Proposition (3 . 5)

Impose p is pure Maximal in M then p is TPM

Proof:-

let p is pure Max submodule of M thus by [11] there exist N⊆ M s.t p ⊆ N ⊆ M , N is pure submodule of M so p is 2-p submodule of M hence p is TPM .

Conversely isn't need success : example: 4Z is TPM of since $z_4 \cong \frac{z}{4z}$ is 2-regular so that every submodule is 2-p submodule but not z_4 regular z-module.

lemma (3 . 6)

impose p is near- Maximal in M then p is TPM .

Proof:-

it's clear by [11] so p is TPM .

Proposition (3 . 7)

Impose M be an R-module and p in M is Two- Maximal then

1. all cyclic submodule in M is TPM in M .
2. all finitely generated submodule in M is TPM in M .

Proof:

1. if p , N are cyclic submodule in M s.t p ⊆ N ⊆ M since every submodule p is Two- Maximal in M , so M/P is 2-regular R-module .Thus by [10] M is 2-regular Rmodule . Hence by [9] all cyclic submodule in M is 2-pure so N is 2-pure in M . so p is TPM in M .
2. Since every submodule is Two-Max so by [10] M is 2-regular Rmodule ,so all finitely generated submodule in M is 2-pure therefore prove become clearly .

Proposition (3 . 8)

Impose M be an R-module then

1. each weak-Maximal of M is TPM .
2. each semi-Maximal of M is TPM .

Proof:-

1. take p , N are submodule and N⊆ M s.t p ⊆ N ⊆ M since every submodule p is weak-Maximal submodule hence M/P is regular Rmodule .Thus by [9] M/P is 2-regular Rmodule so by [10] M is 2-regular Rmodule. Hence every submodule of M is 2-p so N is 2-pure of M . so p is TPM
2. prove clearly since every semi-simple is regular so it's same prove (1) .

Future work

we can in future generalizations these tow concept (TPM- submodule & GEM- ideal) of (fuzzy-Maximal submodule & fuzzy-Maximal ideal) in fuzzy sets .

References

1. Raida D. Mahamood & Mohammed Th. Youns.2010;" On n - Regular Ríngs" College of Computer Sciences and Mathematics University of Mosul.

2. Liang, Z. and Gang, Y. (2007); "On weakly reversible ríngs" Acta Math. Univ. Comenianae,Vol. LXXVI, No. 2, pp. 189-192.
3. Al- Rawi ,H.Y., ((The Prime Spectrum of Regular Ríng)) ph. D. Thesis , Univ. of Mosul, 2004.
4. Anderson E.W. and Fuller K.R., ((Ríngs and Categories of Modules)) , Sprínger – Verlage , New York , 1992 . .
5. Burton D.M., ((First course in Ríngs and Ideals)) , Univ. of New Hamphire , 1970. .
6. Goodreal K.R. ((Von-Nuwmman Regular Ríngs)) Pitman publishing Limited , 1979 .
7. Gooderal K.R., ((Ríng Theory – Non singular Ríngs and Modules)), Marcei – Dekker, New Yourk, 1976.
8. Shwkaea M.R., ((Weak – Mɑximɑl Ideals and Weak – Mɑximɑl Submodules)) M.SC. Thesis, Univ. of Tikrit , 2013.
9. Ghaleb , A.H. ((Generalization on Regular Module and Pure Submodule)) , ph. D. Thesis , Unive. Of Baghdad. 2015.
10. Mohammed D. S., "2-maximal ideals and 2- submodules", Thesis , M.Sc , University of Tikrit, (2017) .
11. Khawla . A. ((Pure Maximal Submodules and Related Concepts)) , Unive of Baghdad. 2023